Chapter 2

EQUILIBRIUM PLASMA PROPERTIES.
OUTER ENVELOPES

After an overall introduction into the neutron star physics, we begin a systematic description of neutron star layers (Fig. 1.2). Chapters 2–4 are devoted to neutron star envelopes. These envelopes are non-uniform. The structure of an envelope for a star with the effective surface temperature $\sim 10^6$ K is schematically shown in Fig. 2.1. We indicate characteristic densities and geometrical depths of interfaces between adjoining layers. One can distinguish the atmosphere, the ocean of ion liquid, followed by solidified layers denoted as the outer and inner crusts. At the bottom of the envelope there may be a mantle containing a liquid crystal of nonspherical nuclei (§ 3.7.2). These layers are already outlined in § 1.3.1 and will be studied in more detail below. The properties of the atmosphere, ocean, and the upper edge of the solidified layer strongly depend on temperature. For the assumed surface temperature $\sim 10^6$ K the internal temperature of a cooling (isolated) star would be $\sim 10^8$ K; the main temperature gradient would occur in the ocean, while deeper layers would be nearly isothermal. When the star cools, the atmosphere and the ocean become thinner. In a sufficiently cold star they may be absent and the solidified matter will extend to the very surface.

Let us remark that neutron star envelopes (which we define as layers containing atomic nuclei or non-uniform nuclear structures, in contrast to the uniform matter in the liquid stellar core) are often called collectively crusts. Strictly speaking, this is inaccurate because the envelopes may contain such layers as the ocean and the mantle, which are not solid. We will mainly use the term “envelope” in this chapter but we will often use the widespread term “crust” (meaning the entire envelope) in subsequent chapters.

The neutron star envelopes are important in many ways. For instance, they provide thermal insulation of the stellar interior and control thus the cooling of
neutron stars (§1.3.7). To study the cooling, one needs to know the equation of state (EOS) and transport properties of the envelopes. The evolution of neutron star magnetic fields is often associated with Ohmic decay of the fields confined in the envelopes. The bursting activity of X-ray bursters (§1.4.6) is explained by nuclear explosions of accreted matter in the outer envelopes. The glitches of radio pulsars (§1.4.4) are commonly associated with depinning of superfluid neutron vortices in the inner crust. Finally, the EOS and opacities of neutron star atmospheres control the spectra of radiation emitted from the surface (e.g., Romani 1987; Pavlov et al. 1995; also see §1.3.1).

While studying the envelopes, one should take into account Coulomb interaction of charged particles (electrons and ions) and strong interaction of nucleons (neutrons and protons in atomic nuclei and free neutron gas between the nuclei in the inner envelope). The importance of the nuclear interaction increases with the growth of the density. In this chapter we focus on the plasma physics effects associated with Coulomb interactions, with the emphasis on the outer envelopes (i.e., for densities lower than the neutron drip density, where Coulomb interactions are most important). Note, however, that many results of the present chapter are valid for the inner envelope. The EOS in the inner envelope (after the neutron drip) will be mainly studied in Chapter 3. Because physical conditions in outer envelopes are similar to those in white dwarfs, the results of the present chapter are equally useful for the physics of white dwarfs.

Here we restrict ourselves to the nonmagnetic envelopes. The equilibrium thermodynamics of charged particles is not affected by the magnetic field\(^1\)

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\(^1\)This statement, proven independently in thesis works by Niels Bohr (Copenhagen, 1911) and by J.H. van Leeuwen (Leiden, 1919; published by van Leeuwen 1921) is known as the Bohr-van Leeuwen theorem. It is often formulated as follows: “At any finite temperature, and in all finite applied magnetic or thermal...
unless the field is quantizing (see §4.1.1). In particular, the results of the present chapter are applicable to the envelopes of those neutron stars whose magnetic field does not exceed $\sim 10^{10} – 10^{11}$ G [more accurate conditions will be given by Eqs. (4.29), (4.30)]. A stronger field can significantly affect thermodynamic functions in the outer envelope. These effects will be considered in Chapter 4.

In §2.1 we introduce the plasma parameters, and in §2.2 describe the basic thermodynamic quantities. Since the conditions of full ionization are satisfied in the largest part of the envelope, we are primarily concerned with the thermodynamics of fully ionized electron-ion plasmas, considered in detail in §2.3 and §2.4. In many applications to neutron stars, one can assume that the electrons constitute an ideal, uniform Fermi gas, and the ions form an one-component plasma (OCP) or a multi-component plasma on the uniform (“rigid”) charge-neutralizing electron background. This case is considered in §2.3. In §2.4 we describe the effects of the Coulomb interactions associated with a compressible electron background under the assumption that the internal structure of bound species is unimportant. Finally, in §2.5 we outline the various approaches to include bound species in the EOS of the stellar atmosphere, and consider partially ionized atomic hydrogen as an example.

2.1. Plasma parameters

2.1.1 Overall parameters

Let $n_e$ be the electron number density and $n_j$ the number density of ion species $j=1,2,\ldots$ (with mass and charge numbers $A_j$ and $Z_j$, respectively). The subscript ‘$j$’ enumerates the ions of different chemical elements or the same element in different ionization states. In the latter case $Z_j$ means the charge number of the atomic nucleus minus the number of bound electrons. The total number density of atomic nuclei (ions) in the plasma is $n_N = \sum_j n_j$. The electric neutrality of the matter implies $n_e = \langle Z \rangle n_N$. Here and in what follows the brackets $\langle \ldots \rangle$ denote averaging over ion species $j$ with statistical weights proportional to $n_j$:

$$\langle f \rangle = \frac{1}{n_N} \sum_j n_j f_j. \quad (2.1)$$

In the OCP, the averaging is unnecessary, so that we shall omit the subscript ‘$j$’ and the brackets whenever possible.
An essential parameter is the number density of baryons, $n_b$. In the outer
envelope, one has $n_b \approx \langle A \rangle n_N$, while in the inner envelope $n_b \approx A' n_N$,
Here $A' \equiv \langle A \rangle + A''$, $\langle A \rangle$ is the average number of nucleons bound in one
nucleus, and $A''$ is the number of free neutrons per one atomic nucleus. We
have $A'' n_N = n_n (1 - w)$, where $n_n$ is the local number density of free neutrons
(in space between atomic nuclei) and $w$ is the fraction of volume occupied by
the atomic nuclei. Clearly, $A' = \langle A \rangle$ in the outer envelope, but $A' > \langle A \rangle$ in
the inner envelope.

The mass density of the matter of relativistic stellar objects is generally
determined as $\rho = \mathcal{E}/c^2$, where $\mathcal{E}$ is the total energy density. However, for
evaluating $\rho$ in the neutron star envelopes it is often sufficient to set $\rho \approx m_u n_b$, where $m_u = 1.6605 \times 10^{-24}$ g is the atomic mass unit.

The characteristic temperatures, which separate the different domains of
plasma parameters, are shown in Fig. 2.2 for typical densities in the outer
envelope (for the inner envelope, see Fig. 3.17).

At low densities and/or high temperatures the matter contains also non-
negligible amount of positrons and photons which contribute to thermodynamic
properties. We mainly neglect their contributions because they are unimportant
for the problems considered in this book. The only exception is §2.5, where
we include the contribution of photons.

### 2.1.2 Electrons

The state of a free electron gas is determined by $n_e$ and $T$. Instead of $n_e$ it
is convenient to introduce the relativity parameter (Salpeter, 1961)

$$x_r \equiv \frac{p_F}{m_e c} \approx 1.00884 \left( \frac{\rho_6 \langle Z \rangle}{A'} \right)^{1/3},$$

(2.2)

where

$$p_F = \hbar k_F = \hbar \left( 3\pi^2 n_e \right)^{1/3},$$

(2.3)

is the electron Fermi momentum and $\rho_6 \equiv \rho/10^6$ g cm$^{-3}$. The electron Fermi
momentum is a measure of electron number density; it limits electron momenta
in the Fermi distribution at $T = 0$. Accordingly, the Fermi energy

$$\epsilon_F = c^2 \sqrt{(m_e c)^2 + p_F^2}$$

(2.4)

is a convenient energy parameter which has the meaning of the chemical po-
tential in a strongly degenerate electron gas. Here and hereafter, we include
the electron rest energy, $m_e c^2$, in the electron energy $\epsilon$ (unless the opposite is
stated explicitly). The electron Fermi temperature is

$$T_F = T_r \left( \gamma_r - 1 \right),$$

(2.5)
Equilibrium plasma properties. Outer envelopes

Figure 2.2. Density-temperature diagram for the outer envelope composed of carbon (left) or iron (right). We show the electron Fermi temperature ($T_F$), the electron and ion plasma temperatures ($T_{pe}$ and $T_{pi}$), the temperature of the gradual gas-liquid transition ($T_1$), and the temperature of the sharp liquid-solid phase transition ($T_m$). Shaded are the regions of typical temperatures in the outer envelopes of middle-aged cooling neutron stars (which are $\sim 10^5 - 10^6$ years old). The lower left domain on the right panel, separated by the dotted line, is characterized by strong electron response or bound-state formation (specifically, it corresponds to the effective charge number $Z_{eff} < 20$ determined as described on p. 91).

where

$$T_r = m_e c^2/k_B \approx 5.930 \times 10^9 \text{ K}$$

(2.6)

is the relativistic temperature unit, $\gamma_r = \sqrt{1 + x_r^2}$, and $k_B$ is the Boltzmann constant. The electron gas is non-relativistic at $T \ll T_r$ and $x_r \ll 1$, and it is ultrarelativistic at $x_r \gg 1$ or $T \gg T_r$. It is nondegenerate at $T \gg T_F$ and strongly degenerate at $T \ll T_F$. For brevity, the density or temperature are called relativistic if the respective parameter $x_r$ or

$$t_r = T/T_r$$

(2.7)

is large.

The Fermi temperature for carbon and iron plasmas at densities $10 \text{ g cm}^{-3} \lesssim \rho \lesssim 10^{10} \text{ g cm}^{-3}$ in the outer envelope is shown in Fig. 2.2. The shaded bands overlap typical temperature profiles of middle-aged cooling neutron stars. At $T \gtrsim 10^7 \text{ K}$, the degeneracy temperature $T_F(\rho)$ is nearly the same for both carbon and iron. However, at $T \lesssim 10^7 \text{ K}$ and $\rho \lesssim 10^4 \text{ g cm}^{-3}$ the curve for iron (on the right panel) bends downward, because the formation of bound species lowers the effective charge of iron ions.
It is also convenient to introduce the electron Fermi velocity \( v_F = \partial \epsilon_F / \partial p_F \) and the corresponding dimensionless parameter

\[
\beta_r = v_F/c = x_r/\gamma_r.
\]  

(2.8)

Sometimes it is useful to introduce the density parameter

\[
r_s \equiv a_e/a_0 = \left( \frac{9\pi}{4} \right)^{1/3} \frac{\alpha_f}{x_r} \approx 0.01400, \]  

(2.9)

where

\[
a_e = \left[ \frac{3}{16\pi n_e} \right]^{1/3}, \]  

(2.10)

is the electron-sphere radius, \( a_0 = \hbar^2/(m_e e^2) \) is the Bohr radius, and \( \alpha_f = e^2/(\hbar c) = 1/137.036 \) is the fine-structure constant. One can write

\[
r_s \approx 1.172 n_{24}^{-1/3}, \]

where \( n_{24} \equiv n_e/10^{24} \text{ cm}^{-3} \). It is also convenient to write

\[
r_s = (\rho_0 s/\rho)^{1/3}, \]

where \( \rho_0 s = 2.696 \text{ g cm}^{-3} \) for the fully ionized hydrogen plasma and \( \rho_0 s = 2.675 (A'/\langle Z \rangle) \text{ g cm}^{-3} \) for the plasma of heavier elements.

The strength of the electron Coulomb interaction in a plasma of nondegenerate electrons can be characterized by the electron Coulomb coupling parameter

\[
\Gamma_e = \frac{e^2}{a_e k_B T}_e \approx 22.75 \frac{T_6}{T_{66}} \left( \rho_6 \langle Z \rangle A' \right)^{1/3}, \]  

(2.11)

where \( T_6 \equiv T/10^6 \text{ K} \). We shall use this parameter \( \Gamma_e \) for any plasma conditions though \( \Gamma_e \) has no transparent physical meaning for degenerate electrons.

In the regions of partially degenerate electrons, one often uses the degeneracy parameter

\[
\theta = T/T_F. \]  

(2.12)

In the non-relativistic and ultrarelativistic cases \( (x_r \ll 1 \text{ and } x_r \gg 1) \), we have

\[
\theta = 0.543 r_s/\Gamma_e \text{ and } \theta \approx (263 \Gamma_e)^{-1}, \]

respectively.

Another important parameter is the screening (Thomas-Fermi) wave number \( k_{TF} \). It is the inverse length \( r_e \) of electron screening in a plasma. In the linear plasma response approximation, a test charge \( q \) embedded in the plasma (at \( r = 0 \)) creates a perturbation \( \tilde{n}_e(r) = (\partial n_e/\partial \mu_e)\tilde{\mu}(r) \) of the electron number density, where \( \tilde{\mu}(r) = -e\phi(r) \) is the perturbation of the electron chemical potential \( \mu_e \) (§ 2.3.1), and \( \phi(r) \) is the excess electrostatic potential determined by the Poisson equation \( \nabla^2 \phi(r) = -4\pi \left[ q\delta^3(r) - e\tilde{n}(r) \right] \). Then we come to the equation \( \nabla^2 + k_{TF}^2 \phi(r) = -4\pi q \delta^3(r) \), with the solution \( \phi(r) = (q/r) \exp(-r/r_e) \), where

\[
r_e^{-1} = k_{TF} = \left( 4\pi e^2 \partial n_e/\partial \mu_e \right)^{1/2}. \]  

(2.13)

\(^2\text{Sometimes the term “Thomas-Fermi wave number” is applied only to } k_{TF} \text{ in the limit of strong degeneracy, Eq. (2.15).}\)
For a nondegenerate electron gas \((T \gg T_F)\) we have \(\partial n_e / \partial \mu_e = n_e / (k_B T)\), and \(r_e\) equals the Debye-Hückel electron screening length (which neglects the contribution of plasma ions, that will be considered below):

\[
r_e \approx a_e / \sqrt{3} \Gamma_e.
\] (2.14)

For a strongly degenerate electron gas \((T \ll T_F)\),

\[
k_{TF} = 2 \sqrt{\alpha_f / (\pi \beta_r)} k_F = \left(0.185 / \sqrt{\beta_r}\right) a_e^{-1}.
\] (2.15)

For further use, it is convenient to introduce the electron plasma temperature

\[
T_{pe} = \hbar \omega_{pe} / k_B \approx 3.300 \times 10^8 x_r \sqrt{\beta_r} \text{K},
\] (2.16)

where

\[
\omega_{pe} = \left(4 \pi e^2 n_e / m^*_e \right)^{1/2}
\] (2.17)
is the electron plasma frequency and \(m^*_e \equiv e_F / c^2 = m_e \gamma_r\) is the effective dynamical mass of an electron at the Fermi surface. In Fig. 2.2, where \(T_{pe}\) is displayed as a function of density in the outer envelope, the corresponding curves for carbon (left panel) and iron (right panel) plasmas at \(\rho \gtrsim 10^3 \text{ g cm}^{-3}\) nearly coincide, because the ratio \(Z/A' \approx \frac{1}{2}\), that determines \(n_e(\rho)\) in Eq. (2.17), is almost the same.

The polarization properties of the electron gas are described by the longitudinal dielectric function \(\varepsilon(k, \omega)\), where \(k\) is the wave number and \(\omega\) is the frequency. Since the electrons respond almost instantly to ion motion, we shall mainly be interested in the static dielectric function \(\varepsilon(k, 0) = \varepsilon(k)\). For the fully degenerate free electron gas, Jancovici (1962) obtained in the random-phase approximation:

\[
\varepsilon(k) = 1 + \frac{k_{TF}^2}{k^2} \left\{ \frac{2}{3} - \frac{2}{3} \frac{y^2 x_r}{\gamma_r} \ln(x_r + \gamma_r) \right. \\
+ \left. \frac{x_r^2 + 1 - 3 x_r^2 y^2}{6 y x_r^2} \ln \left| \frac{1 + y}{1 - y} \right| \right. \\
+ \left. \frac{2 y^2 x_r^2 - 1}{6 y x_r^2} \frac{\sqrt{1 + x_r^2 y^2}}{\gamma_r} \ln \left| \frac{y \gamma_r + \sqrt{1 + x_r^2 y^2}}{y \gamma_r - \sqrt{1 + x_r^2 y^2}} \right| \right\},
\] (2.18)

where \(y = k / (2 k_F)\). In the non-relativistic case \((x_r \ll 1)\), Eq. (2.18) reduces to the Lindhard (1954) dielectric function,

\[
\varepsilon(k) = 1 + \frac{k_{TF}^2}{2 k^2} \left( 1 + \frac{1 - y^2}{2 y} \ln \left| \frac{1 + y}{1 - y} \right| \right).
\] (2.19)
For any $x_T$ and $y \ll 1$, Eq. (2.18) yields
\[
\varepsilon(k) = 1 + \frac{k_{TF}^2}{k^2}.
\]
(2.20)

Equation (2.20) (unlike Eq. (2.18)) is actually valid at any $T$, provided that $\omega$ and $k$ are sufficiently small and $k_{TF}$ is calculated from the general Eq. (2.13) (see, e.g., Chabrier, 1990).

The random-phase approximation does not take into account short range correlations in electron motion. These correlations modify the dielectric function. The modification can be assessed by taking into account the polarization effects in the electron-electron interaction in the density response function. For the non-relativistic electron plasma this correction was described, for instance, by Chabrier (1990):
\[
\varepsilon(k) = 1 + \frac{\varepsilon_L(k) - 1}{1 - (\varepsilon_L(k) - 1)G(k)},
\]
(2.21)
where $\varepsilon_L(k)$ is the Lindhard dielectric function (2.19) and $G(k) = G(k, \omega \to 0)$ is the so-called local field correction which can be evaluated numerically.

At high densities, $x_T \gtrsim 1$, the electrons become relativistic. In this regime, $\varepsilon_F$ is so high that the electron gas is almost ideal and the electron correlations are negligible. Thus, at $x_T \gtrsim 1$ one can use the zero-temperature dielectric function (2.18) in thermodynamic calculations.

At low temperatures, when the ions crystallize (§2.1.3 and 2.3.4), the electron energy spectrum acquires gaps (forbidden energy bands) associated with crystalline structure. The electron band structure is of utmost importance in the solid-state physics (e.g., see, Kittel 1963, 1986) but is relatively tiny and insignificant in high-density stellar matter. However, this structure can be important for some kinetic and neutrino processes in neutron star envelopes (e.g., Raikh & Yakovlev 1982; Kaminker et al. 1999). The effects of electron band structure on thermodynamics of dense stellar matter are almost not explored and will not be discussed below.

### 2.1.3 Ions

At any temperature and density throughout the envelope, the ions are non-relativistic. Their state is determined by Coulomb and nuclear interactions. The strength of the Coulomb interaction of ion species $j$ is characterized by the Coulomb coupling parameter,
\[
\Gamma_j = \Gamma_e Z_j^{5/3} = (Z_j e)^2 / (a_i^{(j)} k_B T),
\]
(2.22)
where
\[
a_i^{(j)} = a_e Z_j^{1/3}
\]
(2.23)
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is the ion sphere radius. In an OCP, we will omit the index ‘\(j\)’. In a multi-component ion plasma, it is useful to introduce the mean ion-coupling parameter \(\Gamma = \Gamma_e \langle Z^{5/3} \rangle\).

At sufficiently high temperatures, the ions form a classical Boltzmann gas. With decreasing \(T\), the gas gradually, without a phase transition, becomes a strongly coupled Coulomb liquid, and then (with a phase transition) a Coulomb crystal. The gas and liquid constitute the neutron star ocean, while the crystal is formed under the ocean (Fig. 2.1).

The gaseous regime of an OCP occurs for \(\Gamma \ll 1\), or \(T \gg T_l\), where

\[
T_l = Z^2 e^2 a_i k_B \approx 2.275 \times 10^7 \frac{\rho_0}{A'} \left(\frac{Z}{\rho_0} \right)^{1/3} K.
\]  

(2.24)

For a plasma of light elements one has \(T_l < T_F\), whereas for heavy elements \(T_l \gg T_F\) (Fig. 2.2).

The ion-sphere radius \(a_i^{(j)}\), defined by Eq. (2.23), is the main ingredient of the ion-sphere model for strongly coupled Coulomb systems \((T \lesssim T_l)\) with nearly uniform electron background. In this model (see, e.g., Shapiro & Teukolsky 1983), the matter is considered as an ensemble of ion spheres filled by the uniform electron background. It is supposed that any ion sphere contains an ion in its center. The radius \(a_i^{(j)}\) is chosen in such a way for the electron charge within the sphere to compensate the ion charge. Then the spheres are electrically neutral and can be treated as non-interacting elements of the ensemble. We will see that the ion-sphere model is very successful in explaining many properties of strongly coupled Coulomb systems.

In the gaseous regime, the characteristic length of the Coulomb screening by ions is the Debye length

\[
r_D = q_D^{-1} = \left[ \frac{4\pi}{k_B T} \sum_j n_j (Z_j e)^2 \right]^{-1/2},
\]  

(2.25)

where \(q_D\) is the Debye wavenumber. In the OCP, \(r_D = a_i/\sqrt{3\Gamma}\).

In the gas of ions \((T \gg T_l)\), the cumulative electron-ion screening wave number is

\[
q_D' = \left( q_D^2 + k_{TF}^2 \right)^{1/2},
\]  

(2.26)

where \(k_{TF}\) is given by Eq. (2.13). If the electrons are non-degenerate and non-relativistic \((T \gg T_F\) and \(x_r \ll 1)\), then \(q_D'\) equals \(q_D\) from Eq. (2.25) with the summation extended over all particle types (ions and electrons).

The scale-length that characterizes quantum-mechanical effects on thermodynamics of liquid and gaseous phases and controls ionization equilibria is the
thermal (de Broglie) wavelength

\[ \lambda_j = \left( \frac{2\pi \hbar^2}{m_j k_B T} \right)^{1/2}, \tag{2.27} \]

where \( m_j \) is the particle mass. In the following, we shall also use the electron thermal length \( \lambda_e \), given by Eq. (2.27) with \( m_j \) replaced by \( m_e \).

As was noted, for instance, by Abrikosov (1960), the ions crystallize at sufficiently high densities. The crystallization of a Coulomb plasma occurs at the temperature

\[ T_m = Z^2 e^2 \frac{a_i k_B}{\Gamma_m} \approx 1.3 \times 10^5 Z^2 \left( \frac{\rho_6}{A'} \right)^{1/3} 175 \Gamma_m K, \tag{2.28} \]

where \( \Gamma_m \) is the melting value of \( \Gamma \). For a classical OCP, \( \Gamma_m \approx 175 \) (§ 2.3.4).

Let us stress the principal difference between the ions and the electrons at low temperatures. Because the electrons are nearly ideal, they become degenerate. In contrast, the ions become strongly coupled by Coulomb forces.

At low \( T \) the quantum effects on ion motion become important. They are especially pronounced at \( T \ll T_{pi} \), where

\[ T_{pi} \equiv \frac{\hbar \omega_{pi}}{k_B} \approx 7.832 \times 10^6 \left( \frac{\rho_6}{A'} \left\langle \frac{Z^2}{A} \right\rangle \right)^{1/2} K \tag{2.29} \]

is the plasma temperature of ions determined by the ion plasma frequency

\[ \omega_{pi} = \left( 4\pi e^2 n_e \left\langle Z^2/m_i \right\rangle \right)^{1/2}. \tag{2.30} \]

The corresponding dimensionless parameter is

\[ t_p = T/T_{pi} \approx 2.92 \left\langle Z \right\rangle^{1/2} \left\langle Z^2/A \right\rangle^{-1/2}/(\Gamma_e \sqrt{x_\pi}). \tag{2.31} \]

In the OCP, \( t_p = \sqrt{R_S/3}/\Gamma \), where

\[ R_S = a_i m_i (Ze)^2/\hbar^2 = r_s (m_i/m_e) Z^{7/3} \tag{2.32} \]

is the ion density parameter, analogous to the electron density parameter \( r_s \) defined by Eq. (2.9). If \( T_{pi} \gg T_m \), huge zero-point ion vibrations suppress crystallization and reduce the melting temperature in comparison with that given by Eq. (2.29). Under these conditions, the actual melting temperature decreases with growing \( \rho \) and drops to zero at a certain \( \rho = \rho_m \) (see § 2.3.4).

The density profiles of the melting and plasma temperatures are plotted in Fig. 2.2 for the outer envelope composed of carbon and iron. For the envelopes composed of the ground-state matter, \( T_m(\rho) \) and \( T_{pi}(\rho) \) will be presented in Fig. 3.17 of Chapter 3. On the left panel of Fig. 2.2 (for carbon), the functions \( T_m(\rho) \)
and \( T_{\text{pi}}(\rho) \) are straight lines (in the logarithmic scale), because carbon is fully ionized in the displayed domain, implying \( Z = 6 \) and \( A = A' = 12 \) in Eqs. (2.28) and (2.29). On the right panel (for iron), these lines bend downward at the lowest displayed densities and temperatures, because of the effective charge lowering.

In a neutron star envelope just beneath the ocean (Fig. 2.1) the crystal can be classical while in deeper layers it is usually in the quantum regime \( (T \ll T_{\text{pi}}) \).

### 2.2. Free energy and thermodynamic quantities

While studying the plasma thermodynamics, it is convenient to choose the volume \( V \) and the temperature \( T \) as independent variables and the Helmholtz free energy \( F(V, T, \{N_j\}) \) as the thermodynamic potential. Here, \( \{N_j\} \) is the set of numbers of various ions \( j \). The free energy is obtained from microscopic physics through the well-known relation (e.g., Landau & Lifshitz 1993)

\[
F(\{N_j\}, V, T) = -k_B T \ln Z(\{N_j\}, V, T),
\]

(2.33)

where \( Z = \text{Tr} \exp(-\hat{H}/k_B T) \) is the canonical partition function of the system, with the total Hamiltonian operator \( \hat{H} \). The equilibrium state can be found by minimizing \( F \) at given \( V \) and \( T \) with respect to the numbers \( \{N_j\} \), subject to stoichiometric constraints. For instance, if a reaction \( A \rightleftharpoons B + C \) (e.g., ionization or dissociation) is allowed in the system, the corresponding stoichiometric relation reads

\[
\partial F/\partial N_A = \partial F/\partial N_B + \partial F/\partial N_C.
\]

(2.34)

The main difficulty is that \( Z(\{N_j\}, V, T) \), as a rule, is not exactly tractable for a real many-body system. Thus one should find such approximations of \( Z \) (or, equivalently, of \( F \)) which, on the one hand, would be calculable and, on the other hand, would preserve the principal physical features of the real system.

One often employs the factorization of the many-body partition function into ideal (corresponding to non-interacting particles) and excess (or configurational) parts. If the Hamiltonian can be separated into the kinetic part \( \hat{K} \) and the potential, momentum-independent part \( \mathcal{U} \), then a translational factor can be extracted from \( Z \). According to Eq. (2.33), this factorization leads to the splitting of the free energy into additive parts:

\[
F(V, T, \{N_j\}) = F_{\text{id}} + F_{\text{ex}} = F_{\text{trans}} + F_{\text{int}} + F_{\text{ex}}.
\]

(2.35)

The internal structure of a composite particle is generally affected by its environment. Hence, the separation (2.35) is approximate. In different layers of the star, it may be applied to different plasma constituents (atoms, molecules, ions, and electrons in the atmosphere; nuclei, free baryons, and leptons in the inner regions). In some layers, one can neglect internal quantum degrees of freedom.
of composite particles (then $F_{\text{int}}$ is dropped) or non-ideality effects (then only $F_{\text{trans}}$ is retained).

The pressure $P$, the internal energy $U$, and the entropy $S$ can be obtained using the thermodynamic relations (e.g., Landau & Lifshitz 1993)

\[
P = -\left(\frac{\partial F}{\partial V}\right)_{T,\{N_j\}}, \quad U = \left(\frac{\partial(F/T)}{\partial(1/T)}\right)_{V,\{N_j\}}, \quad S = -\left(\frac{\partial F}{\partial T}\right)_{V,\{N_j\}}.
\]

Whenever the excess part of the free energy vanishes at high $T$ (which is usually the case), it can be obtained from the internal energy by integrating the second equation,

\[
F_{\text{ex}}(\Gamma, r_s) = \int_{\Gamma_0}^{\Gamma} \frac{U_{\text{ex}}(\Gamma', r_s)}{\Gamma'} d\Gamma'. \quad (2.37)
\]

Second-order thermodynamic quantities are derived by differentiating first-order ones. These are, for example, the logarithmic derivatives of the pressure,

\[
\chi_T = (\partial \ln P / \partial \ln T)_V, \quad \chi_\rho = -(\partial \ln P / \partial \ln V)_T,
\]

and the heat capacities at constant volume or at constant pressure,

\[
C_V = (\partial S / \partial \ln T)_V, \quad C_P = (\partial S / \partial \ln T)_P. \quad (2.39)
\]

An important astrophysical quantity is the adiabatic gradient

\[
\nabla_{\text{ad}} = (\partial \log T / \partial \log P)_S,
\]

that determines the Schwarzschild condition for convective stability (Schwarzschild, 1958); a layer can be convectively unstable if

\[
d \ln T / d \ln P > \nabla_{\text{ad}}. \quad (2.41)
\]

Note that the partial derivatives in Eqs. (2.38)–(2.40) imply a free adjustment of the equilibrium particle numbers $N_j$ to changing conditions. In contrast, the derivatives in Eq. (2.36) assume that these numbers are kept fixed.

From Eqs. (2.38)–(2.40), using the Maxwell relations (e.g., Landau & Lifshitz 1993), we get

\[
C_P = C_V + \frac{PV}{T} \frac{\chi_T^2}{\chi_\rho}, \quad \nabla_{\text{ad}} = \frac{\chi_T}{\chi_T^2 + \chi_\rho C_V T/(PV)}. \quad (2.42, 2.43)
\]

If the free energy is known exactly, different ways to determine second-order quantities [e.g., Eq. (2.40) and Eq. (2.43) for $\nabla_{\text{ad}}$] are equivalent. In numerical models, however, this is not always true. If a numerical EOS does not
satisfy the Maxwell relations in some region of the phase diagram, it is called \textit{thermodynamically inconsistent} in this region. A check for the thermodynamic consistency is an important probe for the accuracy of an EOS (e.g., Saumon \textit{et al.} 1995).

Plasma physics studies are facilitated by the fact that ions are much heavier than electrons. This justifies the \textit{adiabatic (Born-Oppenheimer) approximation}: a configuration of electrons is assumed to be instantaneously adjusted to a given ion configuration, so that the heavy particles (ions, atoms, and molecules) interact via effective (electron-screened) potentials. This approximation will be adopted throughout this chapter.

\subsection*{2.2.1 Fully ionized dense plasma. The basic decomposition}

Neutron star envelopes are mostly fully ionized. This regime can be defined by the requirement that the spacing \( a_i \) between ions is small compared to the Thomas-Fermi radius of the atomic core, \( r_a \sim a_0/Z^{1/3} \), which leads to the condition \( \rho \gg \rho_{\text{eip}} = (m_u/a_0^3)AZ \approx 11 AZ \text{ g cm}^{-3} \) (e.g., Pethick & Ravenhall 1995). At \( \rho \gg \rho_{\text{eip}} \), a model of an \textit{electron-ion plasma} (eip) of bare pointlike nuclei with the charge \( Z = Z_{\text{nuc}} \) on an electron-liquid background can provide a good approximation for thermodynamic functions. Generally, the background is compressible, and Coulomb correlation of ions and electrons should be taken into account. The excess free energy \( F_{\text{ex}} \) in Eq. (2.35) can be represented as a sum of the ionic Coulomb \( F_{\text{ii}} \), the electronic exchange-correlation \( F_{\text{xc}} \), and the ion-electron \( F_{\text{ie}} \) parts. Thus

\begin{equation}
F = F_{\text{id}}^{(i)} + F_{\text{id}}^{(e)} + F_{\text{xc}} + F_{\text{ii}} + F_{\text{ie}}. \tag{2.44}
\end{equation}

To avoid misunderstandings let us stress that the decomposition of \( F \) is not unique. In \( F_{\text{ii}} \) we include the Coulomb interaction between ions as well as the interaction between ions and electrons and between electrons in the approximation of rigid electron background (this uniform negative background compensates the part of \( F_{\text{ii}} \) that goes to infinity in the thermodynamic limit \( N_i \rightarrow \infty \)). Accordingly, \( F_{\text{ie}} \) includes the interaction between the ions and polarized electron background, associated with \textit{deviations} from the rigid background. The term \( F_{\text{xc}} \) includes the exchange and correlation corrections in a purely electron subsystem.

The free-energy decomposition induces decompositions of other thermodynamic quantities. For instance, the pressure can be presented as

\begin{equation}
P = P_{\text{id}}^{(i)} + P_{\text{id}}^{(e)} + P_{\text{xc}} + P_{\text{ii}} + P_{\text{ie}}. \tag{2.45}
\end{equation}

To guide the reader through a subsequent analysis of the different terms, in Table 2.1 we give order-of-magnitude estimates of the various pressure components in a strongly degenerate electron gas (\( T \ll T_F \)) and fully ionized (\( \rho \gg \rho_{\text{eip}} \)), strongly coupled (\( T \ll T_i \)) ion system.
Table 2.1. Order-of-magnitude of the pressure components in Eq. (2.45) for the matter containing strongly degenerate electrons and fully ionized, strongly coupled ions (after Yakovlev & Shalybkov 1989). The parameter $\alpha_B = \alpha_f / \beta_r$ is small at $\rho \gg 1$ g cm$^{-3}$.

| $P_{\text{part}}$ | $|P_{\text{part}}|/P_{\text{id}}^{(e)}$ | Comment |
|------------------|---------------------------------|---------|
| $P_{\text{id}}^{(e)}$ | 1 | Ideal degenerate electron gas, leading term |
| $P_{\text{xc}}$ | $\lesssim \alpha_B$ | Exchange-correlation corrections in electron gas |
| $P_{\text{id}}^{(i)}$ | $\sim T/(ZT_F) \sim \alpha_B Z^{2/3} / \Gamma$ | Ideal ion gas contribution |
| $P_{\text{ii}}$ | $\sim \alpha_B Z^{2/3}$ | Coulomb corrections in the rigid electron background |
| $P_{\text{ie}}$ | $\sim \alpha_B^2 Z^{4/3}$ | Coulomb corrections owing to electron polarization |

The leading contribution comes from the pressure $P_{\text{id}}^{(e)}$ of the ideal degenerate electron gas. Other partial pressures are smaller; their ratios to $P_{\text{id}}^{(e)}$ contain the parameter $\alpha_B = \alpha_f / \beta_r = e^2 / (\hbar v_F)$. It is essentially the fine-structure constant in the ultrarelativistic electron gas, but it grows with decreasing density in the non-relativistic electron gas. The parameter $\alpha_B$ can be regarded as a small parameter of the theory. Its smallness determines the smallness of partial pressures in comparison with $P_{\text{id}}^{(e)}$. Note that contributions of the different terms in second-order thermodynamic quantities can be different than in the pressure. For instance, we will see (§ 2.4.6) that the heat capacity can be determined by ions (by the ii term). In any case the exchange-correlation corrections (xc) and polarization corrections (ei) under the formulated conditions remain to be small. Therefore, we shall drop these corrections in § 2.3. However, because the parameter $\alpha_B$ ceases to be small in a low-density matter (surface layers of neutron stars; partial or removed electron degeneracy; incomplete ionization), all decomposition terms may become important. For this reason we will study the exchange-correlation and polarization corrections in §§ 2.4 and 2.5.

2.3. Fully ionized plasma without electron correlations

If the density is so high that the spacing between electrons is considerably smaller than the $K$-shell radius of an atom, one has a fully ionized plasma of ions immersed in the “rigid” electron background. Then the electron-ion correlations can be neglected. This regime operates at $r_s \lesssim 1/(2Z)$, or $\rho \gtrsim \rho_{\text{rigid}} \approx 22 Z^2 A$ g cm$^{-3}$. The same condition can also be obtained from the requirement for a typical attractive electron-ion Coulomb potential per electron $\sim Ze^2/a_i$ to be much smaller than the electron Fermi energy $\sim \hbar^2/(m_e a_e^2)$. In this section, we assume that the latter condition is fulfilled and consider the plasma of ions immersed in the ideal electron gas.
2.3.1 Ideal electron gas. Fermi-Dirac integrals

2.3.1a Ideal electron gas

The electrons in thermodynamic equilibrium are described by the Fermi-Dirac distribution function

\[ f^{(0)}(\varepsilon - \mu_e, T) \equiv \frac{1}{\exp[(\varepsilon - \mu_e)/k_B T] + 1}, \] (2.46)

where

\[ \varepsilon = \sqrt{m_e^2 c^4 + p^2 c^2} \] (2.47)

is the electron energy and \( p \) is the momentum. It is also convenient to introduce the quantity

\[ \chi = (\mu_e - m_e c^2)/k_B T. \] (2.48)

The free energy of the electron gas can be found from the thermodynamic relation

\[ F^{(e)}_{\text{id}} = \mu_e N_e - P^{(e)}_{\text{id}} V, \] (2.49)

where \( P^{(e)}_{\text{id}} \) is the ideal-gas pressure. The pressure and the number density are, in turn, functions of \( \mu_e \) and \( T \):

\[
P^{(e)}_{\text{id}} = 2k_B T \int \ln \left[ 1 + \exp \left( \frac{\mu_e - \varepsilon}{k_B T} \right) \right] \frac{d^3 p}{(2\pi \hbar)^3} = \frac{8}{3\sqrt{\pi}} \frac{k_B T}{\lambda_e^3} \left[ I_{3/2}(\chi, t_r) + \frac{t_r}{2} I_{5/2}(\chi, t_r) \right], \] (2.50)

\[ n_e = 2 \int f^{(0)}(\varepsilon - \mu_e, T) \frac{d^3 p}{(2\pi \hbar)^3} = \frac{4}{\sqrt{\pi}} \frac{k_B T}{\lambda_e^3} \left[ I_{1/2}(\chi, t_r) + t_r I_{3/2}(\chi, t_r) \right], \] (2.51)

where \( \lambda_e \) is the electron thermal wavelength (2.27), and

\[ I_\nu(\chi, \tau) \equiv \int_0^\infty x^\nu \left( 1 + \frac{\tau x}{2} \right)^{1/2} \exp(x - \chi) + 1 \, dx \] (2.52)

is a Fermi-Dirac integral. The internal energy is given by

\[ \frac{U^{(e)}_{\text{id}}}{V} = \frac{4}{\sqrt{\pi}} \frac{k_B T}{\lambda_e^3} \left[ I_{3/2}(\chi, t_r) + t_r I_{5/2}(\chi, t_r) \right]. \] (2.53)

2.3.1b Fermi-Dirac integrals

In the limit \( t_r \to 0 \), the Fermi-Dirac integrals (2.52) turn into the standard non-relativistic Fermi integrals \( I_\nu(\chi) \), which can be calculated using highly accurate Padé approximations presented by Antia (1993) for \( \nu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \) and \( \frac{5}{2} \).
Table 2.2. Parameters of Eqs. (2.54) and (2.55). The powers of 10 are given in square brackets

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_i^{(0)}$</td>
<td>0.37045057</td>
<td>0.41258437</td>
<td>0.09777982</td>
<td>5.3734153</td>
<td>[−3]</td>
</tr>
<tr>
<td>$c_i^{(1)}$</td>
<td>0.39603109</td>
<td>0.69468795</td>
<td>0.22322760</td>
<td>1.5262934</td>
<td>[−2]</td>
</tr>
<tr>
<td>$c_i^{(2)}$</td>
<td>0.76934619</td>
<td>1.7891437</td>
<td>0.70754974</td>
<td>5.6755672</td>
<td>[−2]</td>
</tr>
<tr>
<td>$\chi_i^{(0)}$</td>
<td>0.43139881</td>
<td>1.7597537</td>
<td>4.1044654</td>
<td>7.7467038</td>
<td>13.457678</td>
</tr>
<tr>
<td>$\chi_i^{(1)}$</td>
<td>0.81763176</td>
<td>2.4723339</td>
<td>5.1160061</td>
<td>9.0441465</td>
<td>15.049882</td>
</tr>
<tr>
<td>$\chi_i^{(2)}$</td>
<td>1.2558461</td>
<td>3.2070406</td>
<td>6.1239082</td>
<td>10.316126</td>
<td>16.597079</td>
</tr>
<tr>
<td>$x_i$</td>
<td>0.07265351</td>
<td>0.2694608</td>
<td>0.533122</td>
<td>0.7868801</td>
<td>0.9569313</td>
</tr>
<tr>
<td>$\xi_i$</td>
<td>0.26356032</td>
<td>1.4134031</td>
<td>3.5964258</td>
<td>7.0858100</td>
<td>12.640801</td>
</tr>
<tr>
<td>$h_i$</td>
<td>0.03818735</td>
<td>0.1256732</td>
<td>0.1986308</td>
<td>0.1976334</td>
<td>0.1065420</td>
</tr>
<tr>
<td>$v_i$</td>
<td>0.29505869</td>
<td>0.32064856</td>
<td>0.07391557</td>
<td>3.6087389</td>
<td>[−3]</td>
</tr>
</tbody>
</table>

The accuracy of non-relativistic formulae decreases rapidly as $T$ goes above $10^7$ K. Fortunately, there exist fully relativistic piecewise fitting formulae for each of the integrals $I_k(\chi, t_r)$ with $\nu = \frac{1}{2}, \frac{3}{2},$ and $\frac{5}{2}$ (Blinnikov et al., 1996; Chabrier & Potekhin, 1998):

\[ I_{k+1/2}(\chi \leq 0.6, t_r) = \sum_{i=1}^{5} c_i^{(k)} \left( \frac{1 + \chi_i^{(k)} t_r/2}{\exp(-\chi_i^{(k)}) + \exp(-\chi)} \right)^{1/2}, \quad (2.54) \]

\[ I_{k+1/2}(0.6 < \chi < 14, t_r) = \sum_{i=1}^{5} \left[ h_i x_i \chi^{k+3/2} (1 + \chi x_i t_r/2)^{1/2} \right] + v_i (\xi_i + \chi)^{k+1/2} (1 + (\xi_i + \chi) t_r/2)^{1/2}, \]  
\[ I_{k+1/2}(\chi \geq 14, t_r) = J_k(\chi, t_r) + \frac{\pi^2}{6} \chi^k \frac{k + 1/2 + (k+1) \chi t_r/2}{R}, \quad (2.56) \]

where $R \equiv \chi^{1/2} (1 + \chi t_r/2)^{1/2},$

\[ J_0(\chi, t_r) = (\chi + t_r^{-1}) R/2 - (2 t_r)^{-3/2} \ln(1 + t_r \chi + \sqrt{2 t_r} R), \quad (2.57) \]

\[ J_1(\chi, t_r) = [2 R^3/3 - J_0(\chi, t_r)]/t_r, \quad (2.58) \]

\[ J_2(\chi, t_r) = [2 \chi R^3 - 5 J_1(\chi, t_r)]/(4 t_r). \quad (2.59) \]

If $\mu_e/(m_e c^2) - 1 \equiv \chi t_r \ll 1,$ then it is advisable to use the non-relativistic limits, $J_k \to \chi^{k+3/2}/(k + 3/2).$ The constants $c_i^{(k)}, \chi_i^{(k)}, x_i, \xi_i, h_i,$ and $v_i$ are listed in Table 2.2. Relative error of Eqs. (2.54)–(2.56) does not exceed 0.2% at $T < 10^{11}$ K (any $\rho$), being typically a few parts in $10^4$. 
Equilibrium plasma properties. Outer envelopes

Table 2.3. Parameters of Eq. (2.61)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i^{(1)}$</td>
<td>44.593 646</td>
<td>11.288 764</td>
<td>39.519 346</td>
<td>-5.751 746</td>
<td>0.265 942 91</td>
</tr>
<tr>
<td>$a_i^{(2)}$</td>
<td>34.873 722</td>
<td>-26.922 515</td>
<td>26.612 832</td>
<td>-20.452 930</td>
<td>11.808 945</td>
</tr>
</tbody>
</table>

2.3.1 c The electron chemical potential

The chemical potential can be found numerically from Eq. (2.51), using Eqs. (2.54)–(2.56). In the nonrelativistic regime, it can also be found from the equation

$$\chi = X_{1/2} \left( \frac{2}{3} \theta^{-3/2} \right),$$

(2.60)

where $\theta$ is the degeneracy parameter and $X_\nu(I)$ is the inverse function to the Fermi integral, also fitted by Antia (1993) for $\nu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. The lowest-order (but still accurate within 0.003%) fit to the function in Eq. (2.60) reads

$$X_{1/2}(z) = \begin{cases} \ln(z R^{(1)}(z)) & \text{for } z < 4 \\ z^{2/3} R^{(2)}(z^{-2/3}) & \text{otherwise,} \end{cases}$$

(2.61)

where

$$R^{(n)}(z) = \frac{a_1^{(n)} + a_2^{(n)} z + z^2}{a_3^{(n)} + a_4^{(n)} z + a_5^{(n)} z^2},$$

and the coefficients $a_i^{(n)}$ are given in Table 2.3.

For arbitrary relativism, the following analytic fit can be used (Chabrier & Potekhin, 1998):

$$\chi = \chi_{\text{nonrel}} - \frac{3}{2} \ln \left[ 1 + \frac{t_r (1 + q_1 \sqrt{t_r} + q_2 q_3 t_r)}{[1 + t_r/(2\theta)](1 + q_2 t_r)} \right].$$

(2.62)

Here, $\chi_{\text{nonrel}}$ is given by the non-relativistic formula (2.60), and the coefficients $q_i$ are functions of $\theta$:

$$q_1 = \frac{3}{2} (e^\theta - 1)^{-1}, \quad q_2 = 12 + 8\theta^{-3/2},$$

(2.63)

$$q_3 = \frac{2}{\pi^{1/3}} - \frac{e^{-\theta} + 1.612 e^\theta}{6.192 \theta^{0.0944} e^{-\theta} + 5.535 \theta^{0.698} e^\theta}. \quad (2.64)$$

The relative error $\delta \chi/\chi$ turns to infinity at $\chi = 0$. However, since thermodynamic quantities are expressed through $\chi$ by virtue of thermal averaging [similar
to that given by Eq. (2.52)], a natural measure of the relative error of the fit is
\[ \delta \chi / \max(\chi, 1) \], Thus determined, the error lies within 0.4% (reached in the range \( 0.7 < \theta < 1.5, T > T_r \)), and it is smaller than 0.2% for \( T < T_r \). Another measure of the inaccuracy is the relative difference between the densities \( n_e \) calculated with the exact and fitted values of \( \mu_e \) in Eq. (2.51). This difference lies within 0.4% at large \( T > T_r \) and within 0.1% at \( T \leq T_r \).

### 2.3.1 The limits of nondegenerate and strongly degenerate electrons

In the limiting case of nonrelativistic classical Boltzmann electron gas (\( T \gg T_F \)) we have \( P_{id}^{(e)} = n_e k_B T \) and \( \chi = \ln(n_e \lambda_e^3/2) \) (the factor 2 takes account of the spin degeneracy).

In the opposite limiting case of very strong degeneracy, \( T \ll T_F \), the Fermi-Dirac integrals are given by Eqs. (2.54)–(2.59). It is important to note that in this case first-order thermodynamic functions depend on \( T \) very weakly. Therefore, the derivatives of \( F_{id}^{(e)} \) over \( T \) cannot be evaluated numerically at \( \theta \gg 1 \) with high accuracy but should be calculated from direct analytic expressions. For example, it is disadvantageous to evaluate the internal energy numerically by taking derivative in Eq. (2.36); the explicit expression (2.53) should be used instead. It is useful to employ the Sommerfeld expansion in powers of \( T^2 \) in order to evaluate the thermodynamic quantities in the regime \( \theta \gg 1 \) (e.g., Yakovlev & Shalybkov 1989). To the first order, the free energy density is

\[
\frac{F_{id}^{(e)}}{V} = \frac{P_r}{8\pi^2} \left[ x_r (1 + 2x_r^2) \gamma_r - \ln(x_r + \gamma_r) - \frac{4\pi^2}{3} t_r^2 x_r \gamma_r \right], \tag{2.65}
\]

where

\[ P_r \equiv \frac{m_e c^2}{\lambda_C^3} = 1.4218 \times 10^{25} \text{ dyn cm}^{-2} \tag{2.66} \]

is the relativistic unit of the electron pressure, and \( \lambda_C \equiv \hbar/(m_e c) = 386.16 \text{ fm} \) is the electron Compton wavelength. The derivatives of Eq. (2.65) calculated using Eqs. (2.36) and (2.39) yield the pressure and specific heat at \( T \ll T_F \):

\[
P_{id}^{(e)} = \frac{P_r}{8\pi^2} \left[ x_r \left( \frac{2}{3} x_r^2 - 1 \right) \gamma_r + \ln(x_r + \gamma_r) \right. \\
+ \left. \frac{4\pi^2}{9} t_r^2 x_r (\gamma_r + \gamma_r^{-1}) \right], \tag{2.67}
\]

\[
C_V^{(e)} = \frac{\pi^2 k_B N_e}{x_r^2} \frac{\gamma_r t_r}{x_r^2} = \frac{k_B^2 T m_e^* P_F V}{3\hbar^3}. \tag{2.68}
\]

Equations (2.65) and (2.67) without thermal corrections (for \( T = 0 \)) were obtained by Frenkel (1928), but his paper remained unnoticed by the astrophysical community. Later these equations were derived independently by Stoner (1930, 1932) and used by Chandrasekhar for constructing models of white dwarfs (e.g., Chandrasekhar 1939).
Equilibrium plasma properties. Outer envelopes

2.3.1 The electron pressure in outer neutron star envelopes and white dwarf cores

In a strongly degenerate electron gas the electron pressure $P_{\text{id}}^{(e)}$ can be accurately approximated by Eq. (2.67) at $T = 0$. In this case $P_{\text{id}}^{(e)}$ is determined only by the electron number density $n_e$. It makes the major contribution to the pressure in degenerate layers of outer envelopes of neutron stars and cores of white dwarfs (see page 66, Table 2.1). Therefore, the EOS and hydrostatic structure of these layers are determined by degenerate electrons. This is especially important for white dwarfs whose overall hydrostatic equilibrium is mainly supported by the electron pressure (e.g., Shapiro & Teukolsky 1983). This “universal” EOS is plotted in Fig. 2.3 as a function of $2\rho Z/A \propto n_e$ (let us remind that $A \approx 2Z$ for astrophysically important atomic nuclei heavier than H).

In the limit of non-relativistic and ultrarelativistic degenerate electrons $P_{\text{id}}^{(e)}$ takes the polytropic form

$$P = K \rho^{\gamma_{\text{ad}}} = K \rho^{1+1/n},$$

(2.69)

where $K$ is a constant, $\gamma_{\text{ad}}$ is the adiabatic index and $n$ is the polytropic index (see, e.g., Shapiro & Teukolsky 1983 for a description of polytropic stars).
More precisely,
\[ P_{\text{id}}^{(e)} \approx P_r x_r^3 3 \gamma_{\text{ad}} / (9 \pi^2 \gamma_{\text{ad}}), \]  
(2.70)
where one has \( \gamma_{\text{ad}} = 5/3 \) (\( n = 3/2 \)) for nonrelativistic electrons (\( x_r \ll 1 \)), and \( \gamma_{\text{ad}} = 4/3 \) (\( n = 3 \)) for ultrarelativistic electrons \( x_r \gg 1 \) (recall that \( x_r \propto \rho^{1/3} \)). In the latter case \( P_{\text{id}}^{(e)} \to (\hbar c/4) (3 \pi^2 n_e^4)^{1/3} \) becomes independent of the electron mass. In the mildly relativistic gas (\( x_r \sim 1, \rho \sim 10^6 \text{ g cm}^{-3} \)) we have \( P_{\text{id}}^{(e)} \sim 10^{23} \text{ dyn cm}^{-2} \) (seven orders of magnitude higher than the pressure in the center of the Sun).

From Fig. 2.3 we see that when the density grows up and the non-relativistic regime (\( x_r \ll 1 \)) is replaced by the ultrarelativistic one (\( x_r \gg 1 \)), the EOS becomes softer. This is important (§ 6.9) for the hydrostatic and thermal structure of outer neutron star envelopes. This softening is even more important for white dwarfs (§ 6.5.3) because it determines the Chandrasekhar limit of their mass (e.g., Shapiro & Teukolsky 1983).

### 2.3.2 Coulomb liquid of ions

#### 2.3.2 a Theory

The free energy of an ideal gas of \( N = N_N \) non-relativistic classical ions is
\[ F_{\text{id}}^{(i)} = N k_B T \left[ \ln(n_N \lambda_i^3 / g_i) - 1 \right], \]  
(2.71)
where \( g_i \) is the spin degeneracy (obviously, the rest-mass energy is not included here). The internal energy and pressure contributions from the ideal gas of ions are simply
\[ U_{\text{id}}^{(i)} = \frac{3}{2} N k_B T, \quad P_{\text{id}}^{(i)} = n_N k_B T. \]  
(2.72)

A study of the excess terms is more complicated. We mark these terms (which arise from the Coulomb interactions assuming a rigid electron background, § 2.2) by the subscript “ii.” In a classical OCP the quantity \( F_{\text{ii}} / N k_B T \) is a function of a single argument \( \Gamma \). This function determines all other excess quantities which can also be expressed via some universal functions of \( \Gamma \) (e.g., Yakovlev & Shalybkov 1989).

In the weak coupling regime (\( \Gamma \ll 1 \)), the theory of Debye & Hückel (1923) is applicable. The Debye-Hückel formula for the free energy of a mixture of non-relativistic weakly coupled ions with charges \( Z_j e \) and number densities \( n_j \) reads (e.g., Landau & Lifshitz 1993, § 78)
\[ \frac{F_{\text{ex}}(\Gamma \to 0)}{V} = -\frac{2e^3}{3} \left( \frac{\pi}{k_B T} \right)^{1/2} \left( \sum_j n_j Z_j^2 \right)^{3/2}. \]  
(2.73)